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New approach for solving master equations of density operators by virtue of the thermal entangled states

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Abstract

Based on the thermal field dynamics, by introducing the thermal entangled state we find a new approach for converting density operators' quantum master equations into c-number equations. This approach, in many cases, is concise and convenient for deriving the solutions of these equations, because the thermal entangled states possess favourable properties.

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1. Introduction

In nature, most systems of interest are coupled to heat reservoirs, so damping and/or losses may be included in the quantum mechanical equation of motion. In the Schrödinger picture, the fundamental problem is to determine how the state, or the density operator, of the system evolves in time. The equation of motion for the density operators has to be solved as a function of time. Such an equation is known as the master equation for the system from which time evolution of the expectation values of system operators may be directly deduced. Using the various quasi-probability representations [1–14] (P-representation, Q-representation, complex P-representation, Wigner functions, etc) for the density operators, quantum master equations are converted into corresponding c-number equations. Each of these quasi-probability representations has its own advantages and shortcomings. In this work we introduce a completely new approach for converting operator master equations into c-number equations in terms of the thermal entangled state representations. As one can see later, the converting processes in many cases are quite direct and concise and the solutions to the new c-number equations are easily derived. Our entangled state representations have been constructed in [15, 16], based on Takahashi–Umezawa thermal field dynamics (TFD) [17].

The work is arranged as follows. In section 2, we briefly review the two thermal state representations and their properties in the context of TFD. One is the coherent thermal state

$\langle\eta|$ [15], and the other is the thermal excitation (and de-excitation) state [16]. In section 3, we convert the master equation of density operator ρ into a c-number equation by projecting the state $\rho|\eta=0\rangle$ onto the $\langle\eta|$ representation. By virtue of the well-behaved properties of $\langle\eta|$, we show that the converting procedures are straightforward and the forms of the c-number equations are neater. Moreover, in many cases these c-number equations are more or less easily solved, and the physical meaning of the solutions become clear. In section 4, we demonstrate how to extract ρ from the solution $\langle\eta|\rho\rangle$.

2. The thermal entangled state

The main point of TFD of Takahashi and Umezawa [17] lies in converting the evaluation of ensemble average at nonzero temperature into the equivalent expectation value with a pure state. This worthwhile convenience is at the expense of introducing a fictitious field (or a so-called tilde-conjugate field). Thus every state $|n\rangle$ in the original real field space $\mathcal{H}$ is accompanied by a corresponding state $|\tilde{n}\rangle$ in $\tilde{\mathcal{H}}$. A similar rule holds for the operators: every operator a acting on $\mathcal{H}$ has an image acting on $\tilde{\mathcal{H}}$. At finite temperature T the thermal vacuum $|0(\beta)\rangle$ ($\beta=\frac{1}{kT}$, k is the Boltzmann constant) is defined by the requirement that the vacuum expectation value agrees with the statistical average, i.e.

$$\langle 0(\beta)|A|0(\beta)\rangle = \mathrm{Tr}(A\,\mathrm{e}^{-\beta H})/\mathrm{Tr}(\mathrm{e}^{-\beta H}) \tag{1}$$

where H is the Hamiltonian. For the ensemble of free bosons with Hamiltonian $H_0=\omega a^\dagger a$, the thermal vacuum state $|0(\beta)\rangle$ is

$$|0(\beta)\rangle = \mathrm{sech}\,\theta\exp[a^\dagger\tilde{a}^\dagger\tanh\theta]|0,\tilde{0}\rangle \qquad \tanh\theta=\exp\left(-\frac{\hbar\omega}{2kT}\right) \tag{2}$$

where the vacuum state $|0,\tilde{0}\rangle$ is annihilated by either a or $\tilde{a}$, $\tanh\theta$ is determined by comparing the Bose–Einstein distribution

$$n=[\exp(\omega\hbar/KT)-1]^{-1} \tag{3}$$

with the expectation value of the thermalized photon number operator $\langle 0(\beta)|a^\dagger a|0(\beta)\rangle$. The relationship between TFD and the squeezed state theory is discussed in [18]. At ultimate high temperature,

$$|0(\beta)\rangle|_{T\to\infty}\to\exp(a^\dagger\tilde{a}^\dagger)|0,\tilde{0}\rangle=\sum_n^\infty|n,\tilde{n}\rangle\equiv|I\rangle. \tag{4}$$

2.1. The coherent thermal state

Based on $|I\rangle$ in [15] we have introduced the coherent thermal state (CTS) $|\eta\rangle$,

$$|\eta\rangle=\exp\left(-\tfrac{1}{2}|\eta|^2+\eta a^\dagger-\eta^*\tilde{a}^\dagger+a^\dagger\tilde{a}^\dagger\right)|0,\tilde{0}\rangle \qquad \eta=\eta_1+\mathrm{i}\eta_2 \quad |\eta=0\rangle=|I\rangle. \tag{5}$$

By respectively operating a and $\tilde{a}$ on $|\eta\rangle$, we have

$$(a-\tilde{a}^\dagger)|\eta\rangle=\eta|\eta\rangle \qquad (\tilde{a}-a^\dagger)|\eta\rangle=-\eta^*|\eta\rangle \tag{6}$$

so $|\eta\rangle$ is the common eigenvector of $(a-\tilde{a}^\dagger)$ and $(\tilde{a}-a^\dagger)$. From

$$\langle\eta|(a^\dagger-\tilde{a})=\eta^*\langle\eta| \qquad \langle\eta|(\tilde{a}^\dagger-a)=-\eta\langle\eta| \tag{7}$$

we get

$$\langle\eta'(a^\dagger-\tilde{a})|\eta\rangle=\eta'^*\langle\eta'|\eta\rangle=\eta^*\langle\eta'|\eta\rangle \qquad \langle\eta'(a-\tilde{a}^\dagger)|\eta\rangle=\eta'\langle\eta'|\eta\rangle=\eta\langle\eta'|\eta\rangle$$

which leads to the orthonormal property

$$\langle\eta'|\eta\rangle = \pi\delta(\eta' - \eta)\delta(\eta'^* - \eta^*). \tag{8}$$

Although the state vector $|\eta\rangle$ is not normalizable in Hilbert space, it can be defined mathematically by means of the rigged Hilbert space in the same way the Dirac is mathematically defined by the unnormalizable eigenstates of position and momentum operators. The completeness relation of $|\eta\rangle$ has been proved in [15],

$$\int \frac{d^2\eta}{\pi}|\eta\rangle\langle\eta| = 1. \tag{9}$$

It then follows from (6) and $\eta = |\eta|\,e^{i\varphi}$ that

$$\sqrt{\frac{a-\tilde{a}^\dagger}{a^\dagger-\tilde{a}}}|\eta\rangle = e^{i\varphi}|\eta\rangle \qquad (a-\tilde{a}^\dagger)(a^\dagger-\tilde{a})|\eta\rangle = |\eta|^2|\eta\rangle. \tag{10}$$

Owing to $[a-\tilde{a}^\dagger, a^\dagger-\tilde{a}] = 0$, they can reside in the same square root. Equation (10) shows that the phase $e^{i\varphi}$ and amplitude $|\eta|$ can be determined at the same time; this contradicts the usual concept that phase and amplitude are not compatible, for example, corresponding to a coherent states $|z = |z|\,e^{i\theta}\rangle$, $a|z\rangle = |z|\,e^{i\theta}|z\rangle$, $\langle z|a^\dagger a|z\rangle = |z|^2$. The operator corresponding to the phase $e^{i\theta}$ is [19, 20]

$$\widehat{e^{i\theta}} = \frac{1}{\sqrt{aa^\dagger}}a$$

and the commutative relation

$$[\widehat{e^{i\theta}}, a^\dagger a] = \widehat{e^{i\theta}}$$

indicates that the phase and amplitude cannot be determined at the same time. But now we have

$$\left[\sqrt{\frac{a-\tilde{a}^\dagger}{a^\dagger-\tilde{a}}}, (a-\tilde{a}^\dagger)(a^\dagger-\tilde{a})\right] = 0$$

so $|\eta = |\eta|e^{i\varphi}\rangle$ might be useful for tackling some density operator equations when the phase and amplitude operators are included.

2.2. *Excitation–de-excitation state [16]*

It is remarkable that the $|\eta\rangle$ state obeys the following equation:

$$(a^\dagger a - \tilde{a}^\dagger\tilde{a})|\eta\rangle = (a^\dagger\eta + \tilde{a}^\dagger\eta^*)\,e^{-\frac{1}{2}|\eta|^2+\eta a^\dagger-\eta^*\tilde{a}^\dagger+a^\dagger\tilde{a}^\dagger}|00\rangle = -i\frac{\partial}{\partial\varphi}|\eta\rangle. \tag{11}$$

Recall in the original Fock space, the number operator $N = a^\dagger a$ acting on the phase state $|e^{i\theta}\rangle$ yields

$$N|e^{i\theta}\rangle = -i\frac{\partial}{\partial\theta}|e^{i\theta}\rangle \tag{12}$$

where [20]

$$|e^{i\theta}\rangle = \sum_{n=0}^{\infty}|n\rangle\,e^{in\theta}. \tag{13}$$

$|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}|0\rangle$ is the eigenstate of N. Comparing (11) with (12) and by analogy with (13), we guess

$$|\eta\rangle = \sum_{E=-\infty}^{\infty} |E, |\eta|\rangle\, \mathrm{e}^{\mathrm{i}\varphi E} \tag{14}$$

where E is an integer. The inverse transformation of (14) is

$$|E, |\eta|\rangle = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} |\eta = |\eta|\,\mathrm{e}^{\mathrm{i}\varphi}\rangle\, \mathrm{e}^{-\mathrm{i}E\varphi}. \tag{15}$$

In fact, from (15) we see that $|E, |\eta|\rangle$ is indeed the eigenstate of $(a^\dagger a - \tilde{a}^\dagger\tilde{a})$,

$$(a^\dagger a - \tilde{a}^\dagger\tilde{a})|E, |\eta|\rangle = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi}\left[\left(-\mathrm{i}\frac{\partial}{\partial\varphi}\right)|\eta = |\eta|\,\mathrm{e}^{\mathrm{i}\varphi}\rangle\right]\mathrm{e}^{-\mathrm{i}E\varphi} = E|E, |\eta|\rangle.$$

On the other hand, from (15) we also have

$$(a - \tilde{a}^\dagger)(a^\dagger - \tilde{a})|E, |\eta|\rangle = |\eta|^2|E, |\eta|\rangle. \tag{16}$$

Equations (15) and (16) coincide with the commutator

$$[(a^\dagger a - \tilde{a}^\dagger\tilde{a}), (a - \tilde{a}^\dagger)(a^\dagger - \tilde{a})] = 0.$$

3. Converting master equations into c-number equations by virtue of the $\langle\eta|$ representation

Let the system's Hamiltonian be H and the density operator be $\rho(a, a^\dagger)$. The system is coupled to a reservoir described by the Hamiltonian H_R; a weak interaction V exists between them. To the second order in the perturbation the evolution equation of the density operator is [1, 2, 5]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}t_1\, \mathrm{Tr}_R[V(t), [V(t_1), \rho_R \otimes \rho(t)]] \tag{17}$$

where Tr_R means the trace over the reservoir variables. In general, there are various ways to convert this operator equation into a c-number equation. Here we use the $\langle\eta|$ representation and the $|I\rangle = |\eta = 0\rangle$ state and recast equation (17) into the form

$$\langle\eta|\frac{\mathrm{d}\rho}{\mathrm{d}t}|I\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle = \frac{1}{\hbar^2}\int_0^t \mathrm{d}t_1\langle\eta|\,\mathrm{Tr}_R[V(t), [V(t_1), \rho_R \otimes \rho(t)]]|I\rangle. \tag{18}$$

In this way the operator equations (17) are transcribed to Schrödinger wavefunction equations in the doubled space. For some systems of H, H_R, V, equation (18) may be easily and completely solved, or partly solved so that some physical information about the solution can be obtained. Without loss of generality, we consider the model operator master equation [4, 5]

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= (\bar{n}+1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \bar{n}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger)\\ &\quad + M(2a^\dagger\rho a^\dagger - a^\dagger a^\dagger\rho - \rho a^\dagger a^\dagger) + M^*(2a\rho a - aa\rho - \rho aa)\end{aligned} \tag{19}$$

which describes a damped generalized harmonic oscillator. Usually, in the earlier literature one used the coherent state and Glauber–Sudarshan P-representation to convert (19) into a Fokker–Planck equation. Here we adopt a completely different approach, i.e. we introduce an auxiliary freedom (tilde Bose operator and tilde states) to define a pure state. This state is just the complete and orthogonal thermal entangled state $\langle\eta|$ which can make up a representative space for the density matrix ρ operating on the state $|I\rangle$. Setting

$$|\rho\rangle = \rho|I\rangle \tag{20}$$

we define a complex function f by

$$f = \langle\eta|\rho\rangle \tag{21}$$

where f contains all information on the density matrix of the physical system. Due to (6), (11) and

$$\langle\eta|\tilde{a} = -\left(\frac{\partial}{\partial\eta} + \frac{\eta^*}{2}\right)\langle\eta| \qquad \langle\eta|a = \left(\frac{\partial}{\partial\eta^*} + \frac{\eta}{2}\right)\langle\eta| \tag{22}$$

we have the rules

$$\begin{aligned}\langle\eta|\tilde{a}|\rho\rangle &= -\left(\frac{\partial}{\partial\eta} + \frac{\eta^*}{2}\right) f \qquad \langle\eta|a|\rho\rangle = \left(\frac{\partial}{\partial\eta^*} + \frac{\eta}{2}\right) f \\ \langle\eta|\tilde{a}^\dagger|\rho\rangle &= \left(\frac{\partial}{\partial\eta^*} - \frac{\eta}{2}\right) f \qquad \langle\eta|a^\dagger|\rho\rangle = -\left(\frac{\partial}{\partial\eta} - \frac{\eta^*}{2}\right) f\end{aligned} \tag{23}$$

$$\langle\eta|(a - \tilde{a}^\dagger)|\rho\rangle = \eta f \qquad \langle\eta|(\tilde{a} - a^\dagger)|\rho\rangle = -\eta^* f \tag{24}$$

$$\langle\eta|(a^\dagger a - \tilde{a}^\dagger\tilde{a})|\rho\rangle = \mathrm{i}\frac{\partial}{\partial\varphi} f. \tag{25}$$

On the other hand, from (4) we also have

$$a|I\rangle = \tilde{a}^\dagger|I\rangle \qquad a^\dagger|I\rangle = \tilde{a}|I\rangle \qquad g(\tilde{a}^\dagger, \tilde{a})\rho = \rho g(\tilde{a}^\dagger, \tilde{a}) \qquad a^\dagger a|I\rangle = \tilde{a}^\dagger\tilde{a}|I\rangle. \tag{26}$$

By operating both sides of operators in (19) on the thermal state $|I\rangle$, and using (26), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle &= \{(\bar{n}+1)(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a}) + \bar{n}(2a^\dagger\tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger) \\ &\quad + M(2a^\dagger\tilde{a} - a^\dagger a^\dagger - \tilde{a}^2) + M^*(2a\tilde{a}^\dagger - aa - \tilde{a}^{\dagger 2})\}|\rho\rangle \\ &= \{(\bar{n}+1)[(a - \tilde{a}^\dagger)\tilde{a} + (\tilde{a} - a^\dagger)a] + \bar{n}[(a^\dagger - \tilde{a})\tilde{a}^\dagger + (\tilde{a}^\dagger - a)a^\dagger] \\ &\quad + M(a^\dagger - \tilde{a})^2 + M^*(\tilde{a}^\dagger - a)^2\}|\rho\rangle.\end{aligned} \tag{27}$$

Then multiplying both sides of (27) by the bra $\langle\eta|$ from the left and using the eigenvector equations (6) and (7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle = (\bar{n}+1)\langle\eta|(\eta\tilde{a} - \eta^* a)|\rho\rangle + \bar{n}\langle\eta|(\eta^*\tilde{a}^\dagger - \eta a^\dagger)|\rho\rangle - (M\eta^{*2} + M^*\eta^2)\langle\eta|\rho\rangle. \tag{28}$$

Using the rules in equations (23) and (24), we can recast the first two terms on the right of (28) to

$$\langle\eta|\{\bar{n}[\eta(\tilde{a} - a^\dagger) + \eta^*(\tilde{a}^\dagger - a)] + (\eta\tilde{a} - \eta^* a)\}|\rho\rangle = \left[-(2\bar{n}+1)|\eta|^2 - \eta\frac{\partial}{\partial\eta} - \eta^*\frac{\partial}{\partial\eta^*}\right]\langle\eta|\rho\rangle. \tag{29}$$

Substituting (29) into (28), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \left[-(2\bar{n}+1)|\eta|^2 - \eta\frac{\partial}{\partial\eta} - \eta^*\frac{\partial}{\partial\eta^*} - M\eta^{*2} - M^*\eta^2\right] f \tag{30}$$

which can be easily connected to the Fokker–Planck equation. Thus our new approach is feasible. One can see that the foregoing derivation is concise and direct, because most of the time in the derivation we were using the eigenvector equation (7), and the rules of (23)–(25) are easily handled. In the following section, we list some examples to show the advantage of working in the $\langle\eta|$ representation, i.e. solution to some c-number master equations can be easily derived.

3.1. Master equation for a damped harmonic oscillator driven by a resonant linear force

The master equation for a damped harmonic oscillator driven by a resonant linear force is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathrm{i}\lambda[a + a^\dagger, \rho] + \frac{g}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a). \tag{31}$$

By operating the both sides of operators on the thermal state $|I\rangle$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle = \left[\mathrm{i}\lambda(a + a^\dagger - \tilde{a}^\dagger - \tilde{a}) + \frac{g}{2}(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a})\right]|\rho\rangle \tag{32}$$

and then projecting equation (32) on $\langle\eta|$ and using (24), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle = \mathrm{i}\lambda(\eta^* + \eta)\langle\eta|\rho\rangle + \frac{g}{2}\langle\eta|(\eta\tilde{a} - \eta^* a)|\rho\rangle. \tag{33}$$

Using (23) and

$$\begin{aligned} &\frac{\partial}{\partial\eta} = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\varphi}\left(\frac{\partial}{\partial r} - \frac{\mathrm{i}}{r}\frac{\partial}{\partial\varphi}\right) \qquad \frac{\partial}{\partial\eta^*} = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\varphi}\left(\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial y}\right) \\ &r \equiv |\eta| \qquad \eta\frac{\partial}{\partial\eta} + \eta^*\frac{\partial}{\partial\eta^*} = r\frac{\partial}{\partial r} \end{aligned} \tag{34}$$

we can further recast (33) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle = \left[2\mathrm{i}\lambda r\cos\varphi - \frac{g}{2}\left(r\frac{\partial}{\partial r} + r^2\right)\right]\langle\eta|\rho\rangle. \tag{35}$$

To search for its solution we factorize $\langle\eta|\rho\rangle = R(r)T(t)$ and substitute it into (35); the result is

$$\frac{T'}{T} - 2\mathrm{i}\lambda\cos\varphi = -\frac{g}{2}\left(r\frac{R'}{R} + r^2\right) = -\mu \tag{36}$$

where μ is the variable-separation constant. From (36) we obtain

$$\langle\eta|\rho\rangle = r^{\frac{2\mu}{g}}\,\mathrm{e}^{-r^2/2}\,\mathrm{e}^{(2\mathrm{i}\lambda\cos\varphi - \mu)t} \tag{37}$$

which involves some relaxation processes. Equation (33) is also convenient for discussing the following particular case, i.e. when $\rho_\alpha = |\alpha(t)\rangle\langle\alpha(t)|$, $|\alpha\rangle = D(\alpha)|0\rangle$ is a pure coherent state, then $a\rho_\alpha = \alpha|\alpha\rangle\langle\alpha|$,

$$\tilde{a}|\rho\rangle_\alpha = |\alpha\rangle\langle\alpha|\tilde{a}|I\rangle = |\alpha\rangle\langle\alpha|a^\dagger|I\rangle = \alpha^*|\rho\rangle_\alpha. \tag{38}$$

Substituting (38) into (33) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle_\alpha = \left[\mathrm{i}\lambda(\eta^* + \eta) + \frac{g}{2}(\eta\alpha^*(t) - \eta^*\alpha(t))\right]\langle\eta|\rho\rangle_\alpha. \tag{39}$$

Especially, when $\alpha(t) = 2\mathrm{i}\frac{\lambda}{g}$, $\alpha^*(t) = -2\mathrm{i}\frac{\lambda}{g}$, (39) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle_{\alpha = 2\mathrm{i}\frac{\lambda}{g}} = 0 \tag{40}$$

which shows that the pure steady state solution of (31) is the pure coherent state $|\alpha(t)\rangle\langle\alpha(t)| = \left|2\mathrm{i}\frac{\lambda}{g}\right\rangle\left\langle 2\mathrm{i}\frac{\lambda}{g}\right|$.

3.2. Model for phase diffusion of a simple harmonic oscillator

Now we consider a model for phase diffusion of a simple harmonic oscillator described by the following master equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\gamma[a^\dagger a, [a^\dagger a, \rho]] = \gamma[2a^\dagger a\rho a^\dagger a - \rho a^\dagger a a^\dagger a - a^\dagger a a^\dagger a\rho]. \tag{41}$$

By operating (41) on the thermal state $|I\rangle$ and multiplying both sides by the bra $\langle\eta|$ from the left, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle = \frac{\gamma}{2}[2a^\dagger a\tilde{a}^\dagger\tilde{a} - (\tilde{a}^\dagger\tilde{a})^2 - (a^\dagger a)^2]|\rho\rangle = -\frac{\gamma}{2}(a^\dagger a - \tilde{a}^\dagger\tilde{a})^2|\rho\rangle. \tag{42}$$

Using rule (25) we can reform equation (42) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta|\rho\rangle = \frac{\gamma}{2}\frac{\partial^2}{\partial\varphi^2}\langle\eta|\rho\rangle. \tag{43}$$

This is a diffusion process of a phase. The solution should be a periodic function of φ with period 2π that remains bounded at $t \to \infty$, thus we choose a simple solution of the form

$$\langle\eta|\rho\rangle = \mathrm{e}^{-\gamma t/2}(A\cos\varphi + B\sin\varphi) \tag{44}$$

where the constants A, B are to be determined by some boundary conditions. Alternately, we can project (42) on $\langle E, |\eta\|$ and using (15) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle E, |\eta\|\rho\rangle = -\frac{\gamma}{2}\langle E, |\eta\|(a^\dagger a - \tilde{a}^\dagger\tilde{a})^2|\rho\rangle = -\frac{\gamma}{2}E^2\langle E, |\eta\|\rho\rangle. \tag{45}$$

Its solution is

$$\langle E, |\eta\|\rho\rangle = \mathrm{e}^{-\gamma E^2 t/2}\langle E, |\eta\|\rho(0)\rangle \tag{46}$$

which shows a rapid decay of amplitude when E is large. Equation (46) agrees with the result of sandwiching (41) in between the number states $\langle n|$ and $|m\rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle n|\rho|m\rangle = \gamma\langle n|[2a^\dagger a\rho a^\dagger a - \rho a^\dagger a a^\dagger a - a^\dagger a a^\dagger a\rho]|m\rangle = \gamma\langle n|\rho|m\rangle(2nm - n^2 - m^2)$$

and

$$\langle n|\rho|m\rangle = \mathrm{e}^{-(n-m)^2\gamma t}\langle n|\rho(0)|m\rangle. \tag{47}$$

4. How to obtain ρ from $\langle\eta|\rho\rangle$

In this section, by taking a concrete example, we show how to recover density operator ρ from $\langle\eta|\rho\rangle$. This example is about the operator master equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = A(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger) + C(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a). \tag{48}$$

As is well known, this equation represents the laser theory in the lowest-order approximation, where A is the gain and C is the cavity loss. According to (6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f = A\langle\eta|(2a^\dagger\tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger)|\rho\rangle + C\langle\eta|(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a})|\rho\rangle \tag{49}$$

where, due to (23) and (24),

$$\begin{aligned}\langle\eta|(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a}) &= \langle\eta|[(\tilde{a} - a^\dagger)a + (a - \tilde{a}^\dagger)\tilde{a}] \\ &= \left[-\left(\eta\frac{\partial}{\partial\eta} + \eta^*\frac{\partial}{\partial\eta^*}\right) - \eta\eta^*\right]\langle\eta|\end{aligned} \tag{50}$$

and

$$\langle\eta|(2a^{\dagger}\tilde{a}^{\dagger} - aa^{\dagger} - \tilde{a}\tilde{a}^{\dagger}) = \langle\eta|[(\tilde{a}^{\dagger} - a)a^{\dagger} + (a^{\dagger} - \tilde{a})\tilde{a}^{\dagger}] = \left[\left(\eta\frac{\partial}{\partial\eta} + \eta^{*}\frac{\partial}{\partial\eta^{*}}\right) - \eta\eta^{*}\right]\langle\eta|. \tag{51}$$

It then follows from (49)–(51) that

$$\frac{\partial f}{\partial t} = \left[(A-C)r\frac{\partial}{\partial r} - (A+C)r^{2}\right]f. \tag{52}$$

The solutions to (52) is

$$f = \langle\eta|\rho\rangle = r^{\frac{\sigma}{A-C}}\exp\left[\frac{A+C}{2(A-C)}r^{2} + \sigma t\right]. \tag{53}$$

The steady state solution to equation (52) is

$$\langle\eta|\rho\rangle = \exp\left(-\frac{r^{2}}{2}\frac{C+A}{C-A}\right). \tag{54}$$

To further obtain the density operator ρ , we multiply both sides of (54) by $\int\frac{\mathrm{d}^{2}\eta}{\pi}|\eta\rangle$ from the left and then employ the completeness relation of $|\eta\rangle$ and (5) to perform the integration

$$\begin{aligned}
|\rho\rangle &= \int\frac{\mathrm{d}^{2}\eta}{\pi}|\eta\rangle\langle\eta|\rho\rangle = \int_{0}^{\infty}\frac{r\,\mathrm{d}r}{\pi}\int\mathrm{d}\varphi\exp\left[-\frac{C}{C-A}r^{2} + r(a^{\dagger}\,\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}\tilde{a}^{\dagger}) + a^{\dagger}\tilde{a}^{\dagger}\right]|0,\tilde{0}\rangle \\
&= \int_{0}^{\infty}\frac{r\,\mathrm{d}r}{\pi}\exp\left(-\frac{C}{C-A}r^{2}\right)\oint\mathrm{d}z\sum_{n=0}\frac{(-r\tilde{a}^{\dagger})^{n}}{\mathrm{i}n!z^{n+1}}\exp(ra^{\dagger}z)|I\rangle \\
&= 2\int_{0}^{\infty}r\,\mathrm{d}r\exp\left(-\frac{C}{C-A}r^{2}\right)\sum_{n=0}\frac{(-r\tilde{a}^{\dagger})^{n}}{(n!)^{2}}(ra^{\dagger})^{n}|I\rangle \\
&= \sum_{n=0}\frac{(-)^{n}a^{\dagger n}a^{n}}{(n!)^{2}}\int_{0}^{\infty}\mathrm{d}r\,r^{n}\exp\left(-\frac{C}{C-A}r\right)|I\rangle \\
&= \sum_{n=0}\left(1-\frac{A}{C}\right)^{n+1}:\frac{(-a^{\dagger}a)^{n}}{n!}:|I\rangle \\
&= \left(1-\frac{A}{C}\right):\exp\left[\left(\frac{A}{C}-1\right)a^{\dagger}a\right]:|I\rangle = \left(1-\frac{A}{C}\right)\exp\left[a^{\dagger}a\ln\frac{A}{C}\right]|I\rangle
\end{aligned} \tag{55}$$

where in the fourth step we have used $a|I\rangle = \tilde{a}^{\dagger}|I\rangle$, and the symbol $:\ :$ denotes normal ordering. Then extracting the state $|I\rangle$ from $|\rho\rangle = \rho|I\rangle$ and comparing with (55), we obtain the density operator

$$\rho = \left(1-\frac{A}{C}\right)\exp\left[a^{\dagger}a\ln\frac{A}{C}\right] = \left(1-\frac{A}{C}\right)\sum_{n=0}^{\infty}(A/C)^{n}|n\rangle\langle n| \tag{56}$$

which obeys $\mathrm{Tr}\rho = 1$. When the laser is below the threshold, $A < C$, by setting $A/C = \mathrm{e}^{-\hbar\omega/KT}$, the photon distribution is

$$\rho_{nn} = (1-\mathrm{e}^{-\hbar\omega/KT})\,\mathrm{e}^{-n\hbar\omega/KT}$$

which behaves as a chaotic state, thus the laser is possible only for $A \geqslant C$, namely, above threshold. From this example, one can see the procedures of obtaining ρ from $\langle\eta|\rho\rangle$. In this respect our formalism is different from the ordinary P-representation, Q-representation and Wigner representation.

Before completing this work we mention some other advantages of using the thermal entangled state representation. (1) We can also apply $\langle\eta|\rho\rangle \equiv f$ to calculate thermal averages

$$\langle A\rangle = \mathrm{Tr}(A\rho) = \sum_{n}^{\infty}\langle n|A\rho|n\rangle = \sum_{n,m}^{\infty}\langle n,\tilde{n}|A\rho|m,\tilde{m}\rangle = \langle I|A\rho|I\rangle = \langle I|A|\rho\rangle$$
$$= \int \frac{\mathrm{d}^2\eta}{\pi}\langle I|A|\eta\rangle\langle\eta|\rho\rangle \tag{57}$$

where in the last step we have used the completeness relation (9). (2) We can also introduce the conjugate state to $|\eta\rangle$, denoted as $|\xi\rangle$, defined as

$$|\xi\rangle = \exp\left(-\tfrac{1}{2}|\xi|^2 + \xi a^\dagger + \xi^*\tilde{a}^\dagger - a^\dagger\tilde{a}^\dagger\right)|0,\tilde{0}\rangle \tag{58}$$

satisfying the eigenvector equations

$$\langle\xi|(a+\tilde{a}^\dagger)|\rho\rangle = \xi\langle\xi|\rho\rangle \qquad \langle\xi|(\tilde{a}+a^\dagger)|\rho\rangle = \xi^*\langle\xi|\rho\rangle \tag{59}$$

and the completeness relation

$$\int \frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi| = 1. \tag{60}$$

Since (59) is different from (24), one may use the $|\xi\rangle$ state to solve some other master equations for other dynamic problems. (3) In the Fock space spanned by the number state $|m,\tilde{n}\rangle$, the state $|\eta\rangle$ can be expanded as

$$|\eta\rangle = \mathrm{e}^{-\frac{1}{2}|\eta|^2}\sum_{m,n=0}^{\infty}\frac{(-1)^n}{\sqrt{m!n!}}H_{m,n}(\eta,\eta^*)|m,\tilde{n}\rangle \tag{61}$$

where $H_{m,n}$ is the two-variable Hermite polynomial, defined as [21]

$$H_{m,n}(\eta,\eta^*) = \sum_{l=0}^{\min(m,n)}\frac{(-1)^l m!n!}{l!(m-l)!(n-l)!}\eta^{m-l}\eta^{*n-l} \tag{62}$$

and we have used the generating function formula of $H_{m,n}$,

$$\exp(-tt' + t\eta + t'\eta^*) = \sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}(\eta,\eta^*). \tag{63}$$

Using equation (61) one can calculate function f from the knowledge of the density operator in the Fock basis.

In summary, in this work based on TFD we have studied the structure of the quantum phase space distributions by virtue of the thermal entangled state representation $|\eta\rangle$ and established a new formalism for obtaining the corresponding c-number master equations, which are the equations for $\langle\eta|\rho\rangle$ or $\langle E,|\eta\|\rho\rangle$. Because the state $|\eta\rangle$ possesses favourable properties (for example, it is the common eigenvector of $(a^\dagger - \tilde{a})$ and $(\tilde{a}^\dagger - a)$), many master equations can be converted into c-number equations much more easily. Moreover, the solutions of the new equations in some cases can be easily derived. We have also shown how to extract ρ from $\langle\eta|\rho\rangle$. The foregoing discussions presented a new approach for solving quantum master equations of density operators which seems concise and convenient for some dynamical systems.

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